EMC Course Notes 2024

Transmission Lines

Transmission Lines

Consider the simple circuit shown in Figure 5.1 modeling a load resistance connected to a source and switch through a pair of wires. The switch turns on at $t = 0$. Applying circuit theory, we expect the voltage across the resistor to change from 0 to V_S at time $t = 0$. However, in real circuits there is always a delay between the time the switch closes and the time the voltage at the load changes. This is because electromagnetic energy travels with a finite velocity.

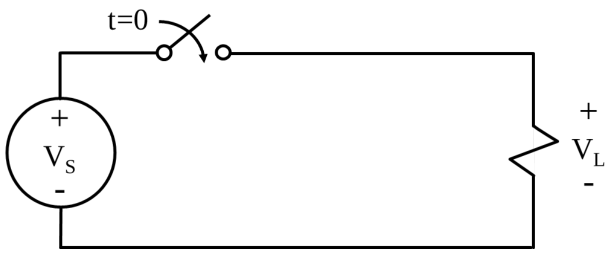


Figure 5.1. A simple circuit.

In high-speed digital circuit designs, there are many situations where the time delay cannot be neglected. Depending on the properties of the dielectric in which the waves travel, the delay associated with circuit board traces or cables is typically on the order of 35-75 picoseconds per centimeter. A signal traveling across a large circuit board or a short cable may arrive several nanoseconds after it was initially sent.

Beyond the obvious timing implications for digital signals, there is an issue related to the fact that the source cannot initially supply the correct current before it has "seen" the load impedance. As a result, signal energy may bounce back and forth between the source and load before reaching the correct steady-state value. This can result in both signal integrity and electromagnetic compatibility problems.

Basic Transmission Line Theory

In situations where the non-zero-time delay is significant, engineers can predict and control the effects of the delay using uniform transmission line theory. Uniform transmission lines are conductor pairs with uniform cross-section that carry electrical power or signals. Several common uniform transmission line configurations are shown in Figure 5.2.

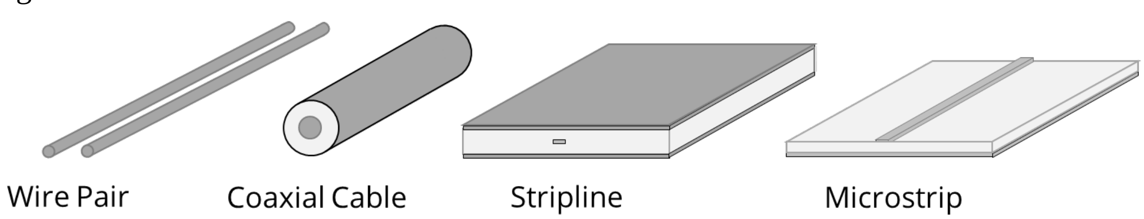


Figure 5.2. Common transmission line configurations.

The cross-sectional dimensions of uniform transmission lines must be small relative to the minimum wavelength contained in the signal. The length of a transmission line, however, is unconstrained and practical transmission lines can be hundreds of wavelengths long. Cross country power lines, telephone wires, TV cables, and high-speed printed circuit board traces are all examples of transmission lines.

In a schematic, transmission lines are typically represented by two parallel rectangles as indicated in Figure 5.3. The symbol resembles a parallel wire pair, but the same symbol is used for any two-conductor transmission line including microstrip traces and coaxial cables. If the transmission line is short and the propagation delay is negligible, it may not have any properties that need to be included in the circuit analysis. For longer conductor pairs, transmission line models can be combined with traditional circuit models to characterize the behavior of the circuit.

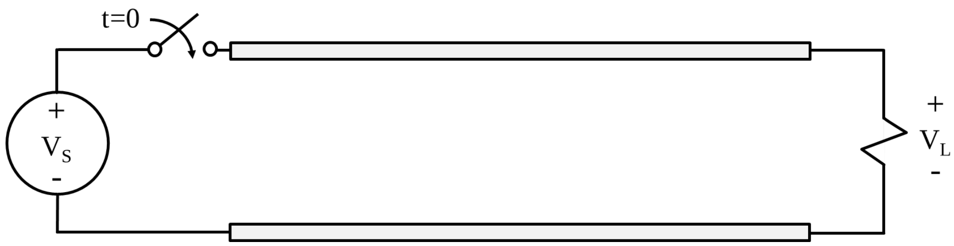


Figure 5.3. A simple transmission line circuit.

Transmission line models recognize that the two conductors have physical properties that may impact their electrical behavior. For example, unless the conductors are super conducting, they will each have a certain amount of resistance. This resistance is not located at a particular point in the circuit, but it is *distributed* along the length of the transmission line. The total resistance of the wires is directly proportional to the length of the line. The *distributed resistance* of the transmission line is the sum of each conductor's per-unit-length resistance and can be expressed in units of ohms per meter (Ω/m).

Current flowing in the conductor pair sets up a magnetic flux in the space surrounding the conductors. Any change in the current amplitude causes a change in the magnetic flux that results in a voltage drop along the length of the line. This voltage can be expressed in terms of an inductance times the rate of change of the current, just as in circuit theory. This inductance is not located at one point, however. It is distributed along the length of the transmission line and can be expressed in units of henries per meter (H/m).

Transmission lines also have a distributed capacitance (farads per meter, F/m) due to the electric field coupling between the two conductors. If there is a dielectric material surrounding the conductors, current leaking from one conductor to the other through the dielectric can be represented by a distributed conductance (siemens per meter, S/m).

The equivalent circuit shown in Figure 5.4 models the effects of the distributed resistance, inductance, capacitance, and conductance. In this circuit representation, the *distributed parameters* of the transmission line are modeled using *lumped elements*. The lumped element model divides the transmission line into several electrically short sections. The inductance of each section is represented by an inductor; whose value is the inductance per unit length of the transmission line times the length of the section. A series resistor, capacitor, and parallel resistor similarly represent the resistance, capacitance, and conductance of each section, respectively. Note that the resistors in Figure 5.4 represent the sum of the resistances per unit length of both conductors. The inductors represent the inductance per unit length of the conductor pair. They are combined for simplicity, but it's important to recognize that neither conductor in the actual transmission line is a perfectly conducting equipotential surface.

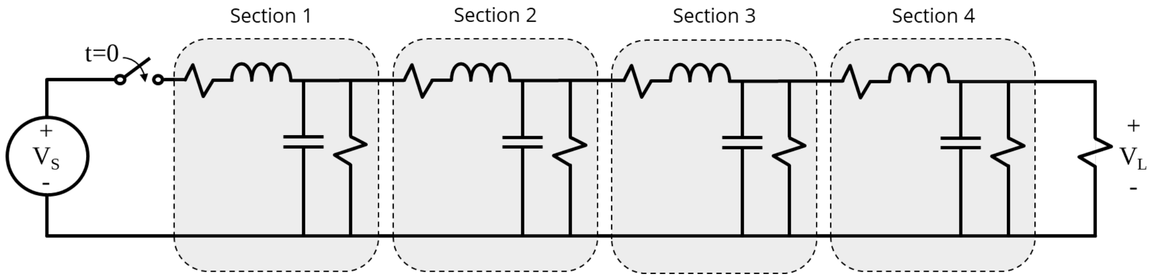


Figure 5.4. Lumped-element model of transmission line circuit.

The Transmission Line Equations

Consider the circuit representation for a single section of transmission line shown in Figure 5.5. *V(z,t)* represents the time-varying voltage at position *z* between the two conductors. The current on one of the conductors at position *z* is represented as *I(z,t)*. Note that the currents on the two conductors of a transmission line at a given position along its length are always opposite in sign and equal in amplitude. Therefore, it is not necessary to represent the current on each conductor independently.

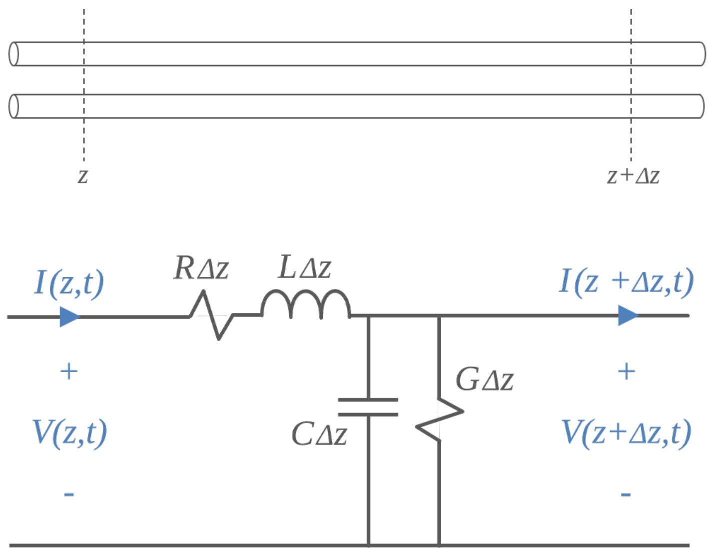


Figure 5.5. Equivalent circuit for a section of transmission line.

Applying Kirchhoff's voltage law to the model in Figure 5.5, we find that the difference between the potential at position z and position $z+\Delta z$ is given by,

$$V(z,t) - V(z+\Delta z,t) \approx \Delta z\left[RI(z,t) + L\frac{\partial I(z,t)}{\partial t}\right]. \qquad (5.1)$$

Since Δz is small[1], $V(z+\Delta z, t)$ may be represented by its Taylor series expansion,

[1] relative to the time rate-of-change of *V(z,t)* and *I(z,t)*, a condition enforced by our requirement that each section of the lumped element model be electrically short.

$$V(z+\Delta z,t) \approx V(z,t) + \Delta z \frac{\delta \partial V(z,t)}{\partial z} + \cdots \quad . \tag{5.2}$$

Neglecting the higher order terms and substituting (5.2) into (5.1) we get,

$$\frac{\partial V(z,t)}{\partial z} = -\left[L\frac{\partial I(z,t)}{\partial t} + RI(z,t) \right]. \tag{5.3}$$

Although this is an approximation, it can be made to be arbitrarily accurate by choosing Δz (the length of each section) to be sufficiently small.

The difference in the current at the two positions, z and $z+\Delta z$, must be equal to the current leaked through the conductance $G\Delta z$ and the capacitance $C\Delta z$,

$$I(z,t) - I(z+\Delta z,t) \approx \Delta z \left[GV(z,t) + C\frac{\partial V(z,t)}{\partial z} \right]. \tag{5.4}$$

Using the Taylor series expansion as before, we obtain,

$$\frac{\partial I(z,t)}{\partial z} = -\left[C\frac{\partial V(z,t)}{\partial t} + GV(z,t) \right]. \tag{5.5}$$

Equations (5.3) and (5.5) are coupled differential equations governing the current and voltage in transmission lines and are often referred to as the *telegrapher's equations*.

Lossless Transmission Lines

If the resistance per unit length, R, and the conductance per unit length, G, of a transmission line are negligible (i.e., $R \approx G \approx 0$), then the transmission line is lossless. For a lossless line, the transmission line equations (5.3) and (5.5) reduce to,

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t} \tag{5.6}$$

and

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}. \tag{5.7}$$

Differentiating (5.6) with respect to z and (5.7) with respect to t and combining the two equations, we get an equation describing the voltage on the transmission line as a function of time and position,

$$\frac{\partial^2 V}{\partial z^2} - LC\frac{\partial^2 V}{\partial t^2} = 0. \tag{5.8}$$

Similarly, differentiating (5.6) with respect to t and (5.7) with respect to z results in an equation for the current on the transmission line,

$$\frac{\partial^2 I}{\partial z^2} - LC\frac{\partial^2 I}{\partial t^2} = 0. \tag{5.9}$$

Equations that relate a second derivative in time to a second derivative in space, in the way that Equations (5.8) and (5.9) do, are *wave equations*. Solutions to a wave equation are linear combinations of traveling wave functions. For example, the general solution to Equation (5.8) can be written in the form,

$$V(z,t) = F_1(t - \frac{z}{v}) + F_2(t + \frac{z}{v}) \quad (5.10a)$$

where,

$$v = \frac{1}{\sqrt{LC}} \quad (5.10b)$$

and F_1 and F_2 represent arbitrary functions. The nature of the functions F_1 and F_2 is determined by the source exciting the transmission line and the load terminating the transmission line.

Traveling Waves

Solutions of the form $F_1(t - \frac{z}{v})$ are referred to as traveling waves. Consider the rectangular pulse function given by,

$$F_1(t - \frac{z}{v}) = \begin{cases} A & 0 \leq t - \frac{z}{v} \leq \tau \\ 0 & \text{otherwise.} \end{cases} \quad (5.11)$$

This function is plotted in Figure 5.6 as a function of z for four different values of t. Note that as t increases, the function shifts in the z direction. The pulse effectively propagates in the z direction. The velocity of this propagation is the change in distance divided by the change in time or

$$v = \frac{\Delta z}{\Delta t}. \quad (5.12)$$

Therefore, $F_1(t - \frac{z}{v})$ represents a wave traveling with a velocity, v, in the z direction.

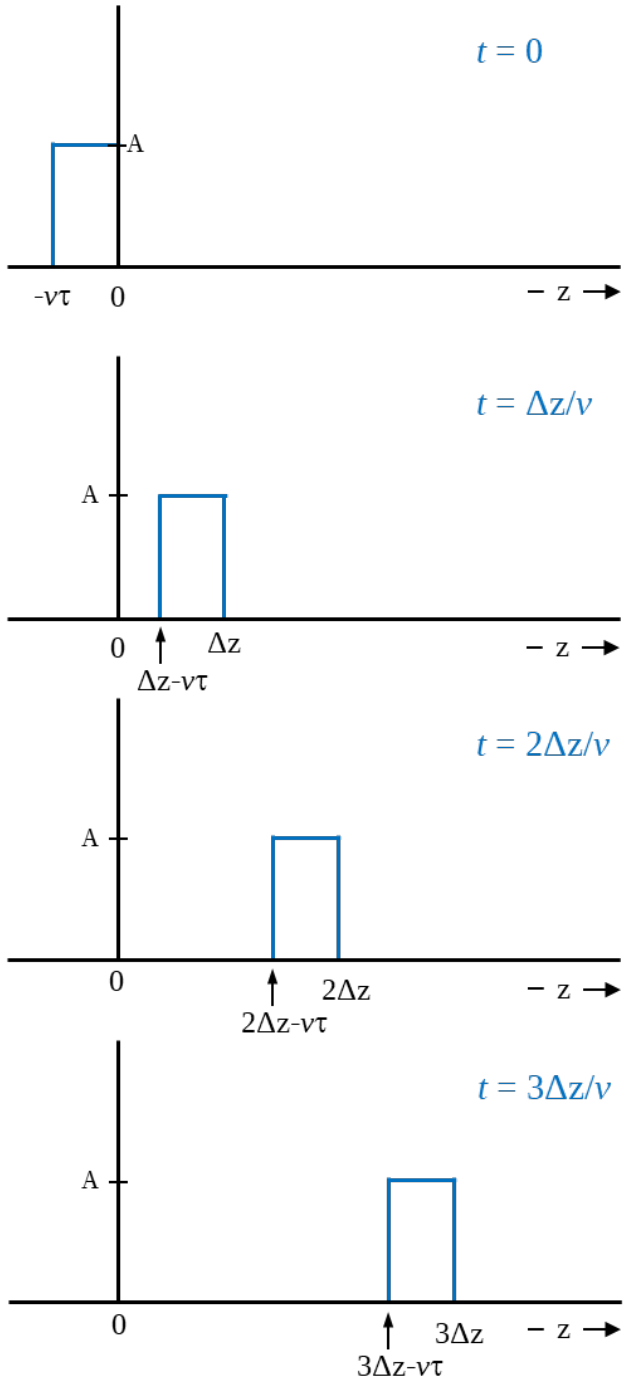


Figure 5.6. Pulse function propagating in *z* direction.

Semi-Infinite Transmission Lines

Consider a semi-infinite line extending from $z = 0$ to $z \rightarrow \infty$ as illustrated in Figure 5.7. Before the switch is closed, there is no charge on (and therefore no voltage across) the line. At $t = 0$, the switch is closed causing a voltage V_0 to appear at the input to the transmission line. The voltage V_0 does not appear instantly at all points along the line. Instead, a wave of voltage progresses along the line. The farther a given point is from the voltage source, the later the time at which the line voltage at that point jumps from 0 to V_0. Charge flowing onto the line results in a current. A current wave travels from the voltage source in step with the voltage wave. The current, I_0, on the two wires is equal in amplitude and opposite in direction at any given distance along the line at any time.

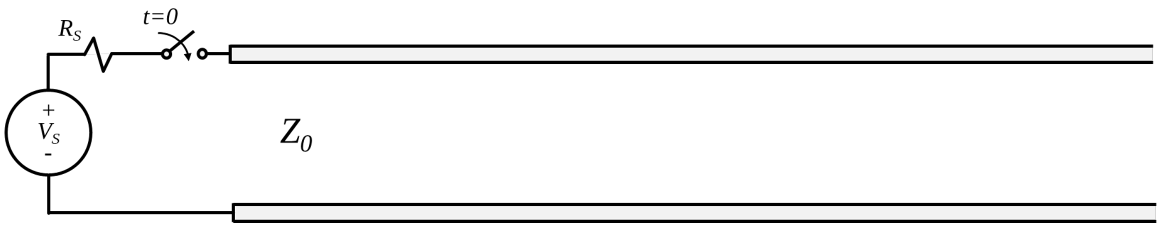


Figure 5.7. Semi-infinite transmission line.

The voltage wave progresses only as fast as the line current can carry charges to the wave front to produce the change in voltage. The current wave can only be supported on parts of the line where sufficient voltage exists to force the movement of charge. In other words, the voltage and current wave fronts must move along the line together.

The characteristic impedance, Z_0, of a transmission line is defined as the ratio of the voltage to the current in a forward-traveling wave. Suppose that at time $t = t_0$ the wave front is located at $z = z_0$, and at $t = t_0+\Delta t$, the wave front is located at $z = z_0+\Delta z$. During the time interval Δt, the total charge flowing out of the voltage source is $I_0\Delta t$. During the same time interval, the voltage across a capacitance $C\Delta z$, associated with a short section Δz of the line, is raised from 0 to V_0. This is accomplished by storing charge equal to $V_0\, C\, \Delta z$ on that section of the line. Thus,

$$I_0 \Delta t = V_0 C \,\Delta z \,. \tag{5.13}$$

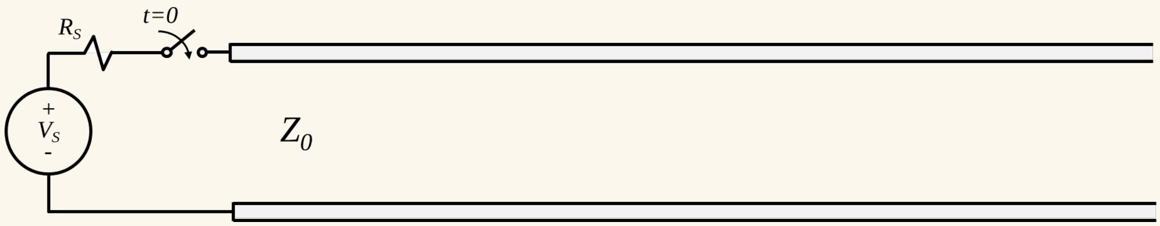
From (5.10b), the velocity of propagation, $\Delta z/\Delta t$, is $1/\sqrt{LC}$, so the characteristic impedance is,

$$Z_0 = \frac{V_0}{I_0} = \frac{\Delta t}{C \Delta z} = \frac{\sqrt{LC}}{C} = \sqrt{\frac{L}{C}} \,. \tag{5.14}$$

The characteristic impedance has the same units as a resistance (ohms). The product of the voltage across and the current through a resistor represents power that is lost as heat. The product of the voltage across and the current through a real-valued characteristic impedance represents power that is being carried away. It is lost to the source, but still remains in the form of electromagnetic energy flowing down the transmission line.

Example 5-1: Semi-infinite Transmission Line

Find an expression for the voltage and the current on the semi-infinite transmission line illustrated below.



A DC source with an open circuit voltage V_s and source impedance, R_s, is connected to the transmission line through a switch that closes at time $t = 0$. At the instant that the switch first closes, a voltage appears across the input to the transmission line and a current starts to flow. By Kirchhoff's current law, the current into the transmission line, I_0, is equal to the current delivered by the source.

By Kirchhoff's voltage law, $V_S = I_0 R_S + V_0$. Or, since $V_0 = I_0 Z_0$, $I_0 = \frac{V_S}{R_S + Z_0}$.

In other words, the input to the transmission line looks like a resistor with impedance Z_0 when the switch is first closed. Once the current starts to flow onto the transmission line, a current and voltage wave front propagate down the line with velocity, v. Since the line is infinitely long, this wave can propagate forever, and the line draws a constant current from the voltage source. From the point of view of the source, this transmission line behaves exactly like a resistor with a resistance equal to Z_0.

In terms of the general solution to the wave Equation (5.10), the arbitrary function $F_2(t+\frac{z}{v})$ must be equal to zero, since there is no mechanism to generate a wave going in the $-z$ direction. At the input to the transmission line,

$$V(0,t) = \begin{cases} 0 & t < 0 \\ V_S \dfrac{Z_0}{R_S + Z_0} & t \geq 0 \end{cases}.$$

Therefore, $F_1(t)$ is equal to the unit step function U(t) and the solution for $V(z,t)$ must be,

$$V(z,t) = \mathrm{U}(t - \frac{z}{v})$$

and $I(z,t)$ is given by, $I(z,t) = \frac{1}{Z_o} \mathrm{U}(t - \frac{z}{v})$.

Finite Length Transmission Lines

A finite length transmission line behaves exactly like a semi-infinite transmission line during the first few moments after the excitation appears at the input. After all, the transmission line's termination has no way of affecting the signal at the input until some of the input energy has propagated down to the termination point and back to the observation point.

Consider the transmission line with an open-circuit termination shown in Figure 5.8. As the switch is closed at time $t = 0$, current is supplied by the voltage source to charge the line and a wave is launched. Since this wave travels in the positive z direction, it will be referred to as a forward-traveling or incident wave. Until the wave reaches the termination at $z = \ell$, this circuit behaves exactly as if the transmission line was semi-infinite. However, when the incident wave reaches the open circuit end of the transmission line, the current is suddenly forced to go to zero. Back at the source end of the line, current continues to flow even after the incident wave reaches the termination, since it takes a finite amount of time before the nature of the termination can be conveyed back to the source. In order to meet the boundary conditions at the termination, a new wave is launched from the termination end. This is a backward-traveling or reflected wave. At all times, the sum of the incident and reflected waves must meet the boundary condition imposed by the termination. In this case, since the termination is an open circuit, the current in the reflected wave must be the negative of the current in the incident wave at $z = \ell$. In other words,

$$I^{+}\left(t-\frac{\ell}{v}\right)=-I^{-}\left(t+\frac{\ell}{v}\right) \tag{5.15}$$

where $I^{+}(z,t)$ is the incident wave and $I^{-}(z,t)$ is the reflected wave.

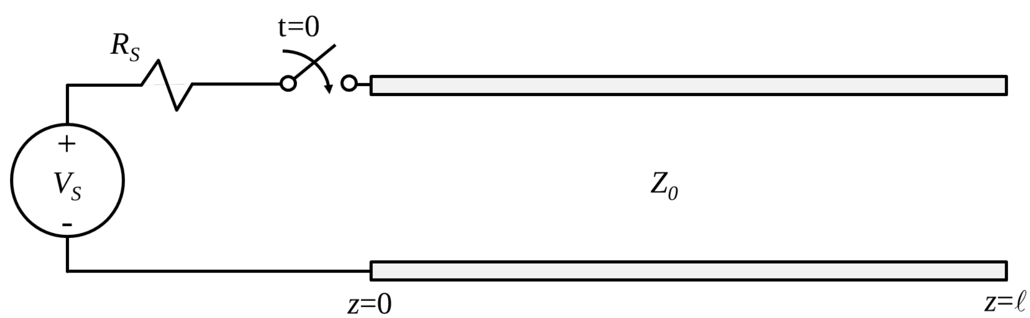


Figure 5.8. Finite length transmission line.

There is also a voltage associated with the reflected wave. The ratio of voltage to current in a reflected (-z directed) wave is equal to $-Z_0$.[1] Therefore, if the current in the incident wave was I_0 and the voltage was V_0, then the current in the reflected wave is $-I_0$ and the voltage is still V_0. At an open circuit termination, the current goes to zero and the voltage doubles.

[1] The sign of Z_0 indicates which direction the power is flowing relative to the direction arbitrarily defined as the direction of positive current flow.

Now consider the finite length transmission line terminated in a resistance R_L shown in Figure 5.9. When the incident wave reaches the termination, the ratio of the voltage to the current is forced to equal R_L. Unless $R_L = Z_0$, this requires a reflected wave to be set up in order to meet this condition.

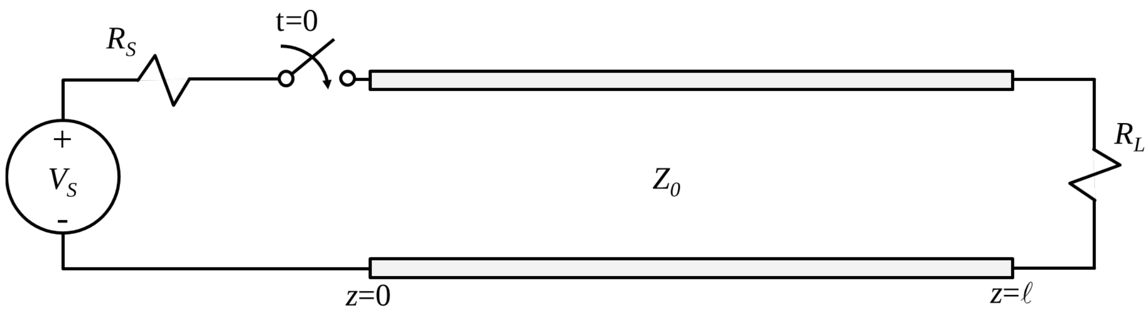


Figure 5.9. Terminated transmission line.

Let's let V^+ and I^+ represent the amplitude of the incident voltage and current waves, respectively. V^- and I^- will represent the amplitudes of the reflected voltage and current waves where,

$$\frac{V^+}{I^+} = Z_0 = -\frac{V^-}{I^-}\,. \tag{5.16}$$

When the incident wave reaches the point $z = \ell$, the total voltage becomes $V^+ + V^-$ and the total current becomes $I^+ + I^-$. By Ohm's law, the ratio of the total voltage to the total current must be R_L at $z = \ell$. Therefore,

$$\frac{V^+ + V^-}{I^+ + I^-} = \frac{V_L}{I_L} = R_L\,. \tag{5.17}$$

Combining Equations (5.16) and (5.17) we get,

$$Z_o\,\frac{V^+ + V^-}{V^+ - V^-} = R_L\,. \tag{5.18}$$

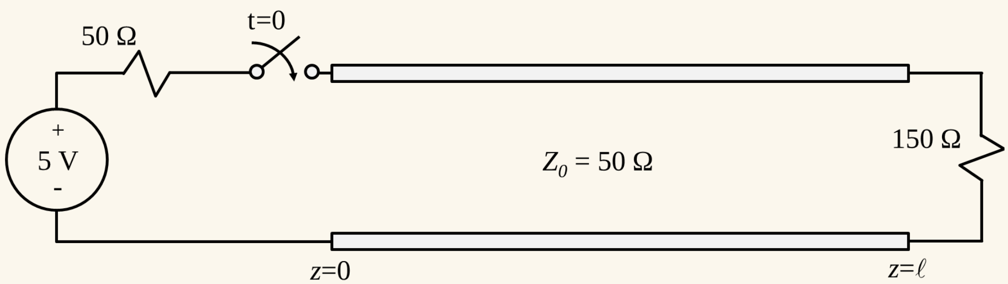
We can solve this equation to get the ratio of the reflected voltage to the incident voltage, which is also known as the *voltage reflection coefficient*, Γ.

$$\Gamma = \frac{V^-}{V^+} = \frac{R_L - Z_o}{R_L + Z_o}\,. \tag{5.19}$$

This reflection coefficient is an important and useful concept in transmission line theory. Note that when $Z_0 = R_L$, $\Gamma = 0$. In other words, if a transmission line is terminated with a resistance equal to its characteristic impedance, there is no reflected wave. When the termination resistance is zero (short circuit), $\Gamma = -1$. When the termination resistance is infinite (open circuit), $\Gamma = 1$. The magnitude of the reflection coefficient is always less than or equal to 1.

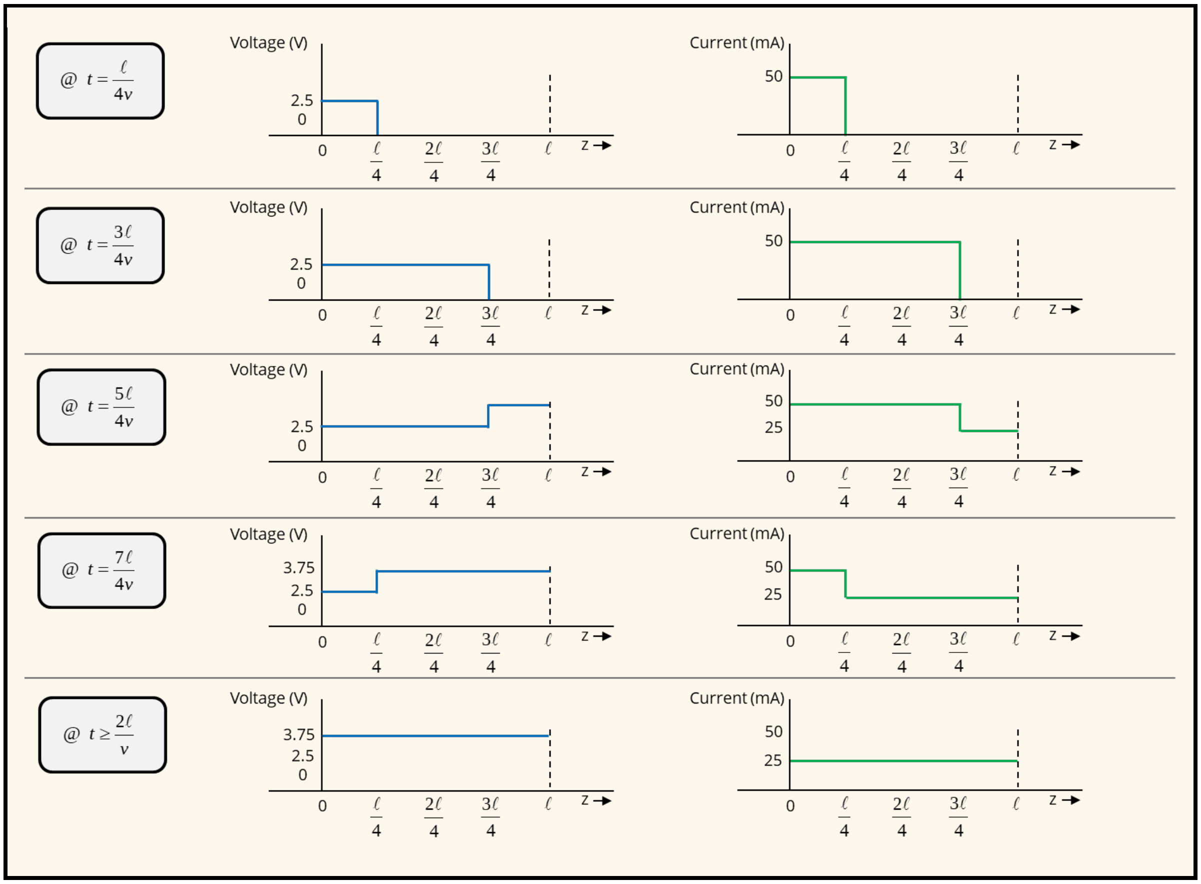
Example 5-2: Terminated Transmission Line

Plot the voltage and current distribution along the line shown in the figure below.



At t = 0+ (i.e., right after the switch is closed), the impedance presented to the source is the series combination of Rs and Z_0. Thus, the current drawn from the source is 5/(50+50) = 50 mA and the voltage across the line at $z = 0$ is 2.5 volts. The incident voltage and current waves propagate down the line with a speed of v. At $t = (\ell/4v)$, the wave front is at $z = \ell/4$, as shown in the figure below. Note that up to this point the load resistance has no effect at all. At $t = 5\ell/4v$, the voltage and current wave have reflected off the termination. The voltage reflection coefficient, Γ, is (150-50)/(150+50) = ½. Therefore, the total voltage on the parts of the transmission line that have seen both the incident and reflected waves is 2.5 + 1.25 = 3.75 volts. The reflected current is –1/2 x 50 = -25 mA. Therefore, the total current on the parts of the line that have seen both the incident and reflected waves is 50 – 25 = 25 mA.

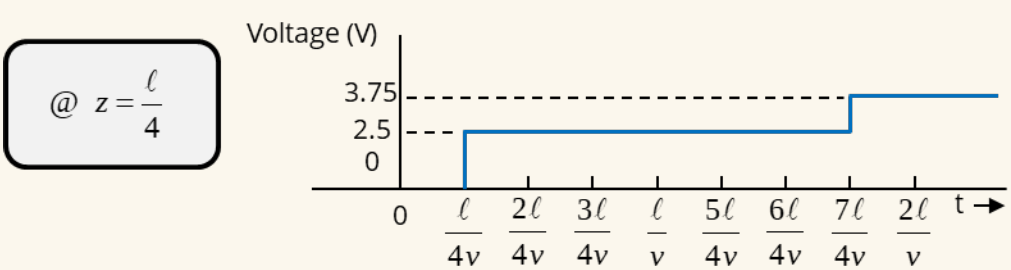
At $t = 2\ell/v$, the reflected wave reaches the source. If the source impedance was not equal to the characteristic impedance of the transmission line, another reflection would take place, and a second forward-traveling wave would be launched. In this example however, the source impedance and characteristic impedance are equal. Γ = 0 at this end of the line and there is no reflection. For $t > 2\ell/v$ there is no change in the voltage and current on the line. The system has reached a steady-state solution.



Example 5-3: Transmission Line Voltage as a Function of Time

Plot the voltage as a function of time when measured at a position $z = \ell/4$ along the transmission line in Example 5-2.

The figure below shows the voltage as a function time as it would be measured by a high-impedance oscilloscope located at a position $z = \ell/4$. Note that even though the switch closes at time $t = 0$, nothing is observed until the incident wave passes by the oscilloscope position at time $t = \ell/4v$. There is then no further change in the voltage until the reflected wave passes by at time $t = 7\ell/4v$.



The Bounce Diagram

Note that the voltage at any given position at any given time is the superposition of the voltages of every wave that has passed by that position up to that time. If both the source and the load impedance are not equal to the characteristic impedance, reflections occur at both ends of the line. In this situation, waves will bounce back and forth forever. However, since the reflection coefficient is always less than or equal to one, each reflected wave is smaller than the waves that preceded it and the solution will converge to a steady state value.

Solving for the voltage and current waveforms on a transmission line that is mismatched at both ends can be very complicated. A *bounce diagram* is a simple graphical means of keeping track of multiple reflections. A bounce diagram is a space-time diagram with distance plotted horizontally along a line from the input to the load while time is plotted vertically.

A bounce diagram for the transmission line in Examples 5-2 and 5-3 is shown in Figure 5.10. The locus of the wave front is represented by the end of a line that bounces back and forth, while moving down the face of the diagram. Since the wave front travels at constant speed, the angle of descent is constant, and the wave front traces a zigzag line as it bounces between the source and load. The amplitude of each wave front is written directly above each section of the line. The amplitude of a wave front is determined by multiplying the amplitude of the wave front above it by the reflection coefficient of the last termination it encountered.

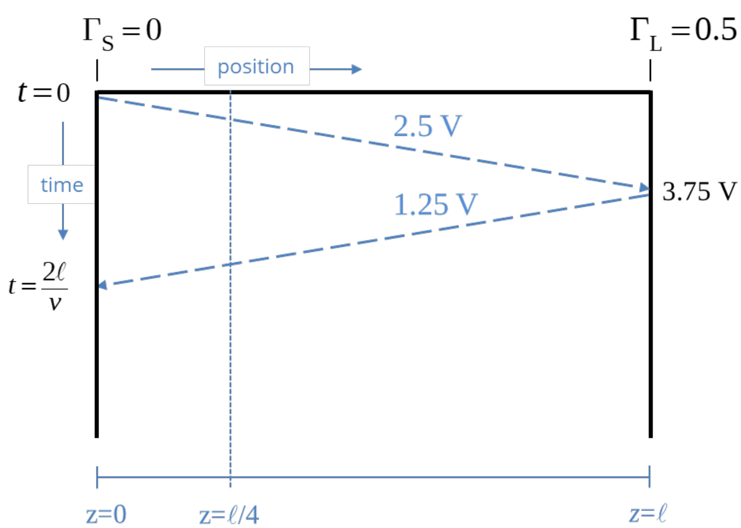
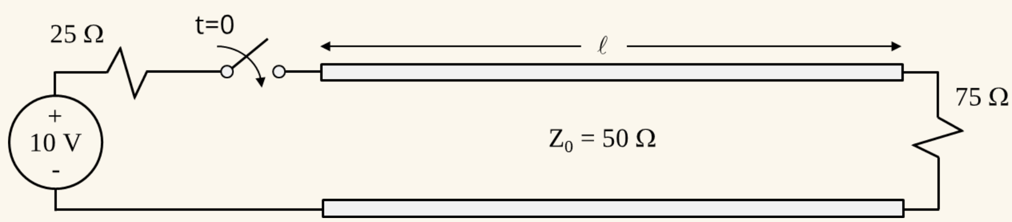


Figure 5.10. Bounce diagram for Examples 5-2 and 5-3.

Example 5-4: Transmission Line Voltage and Current as a Function of Time

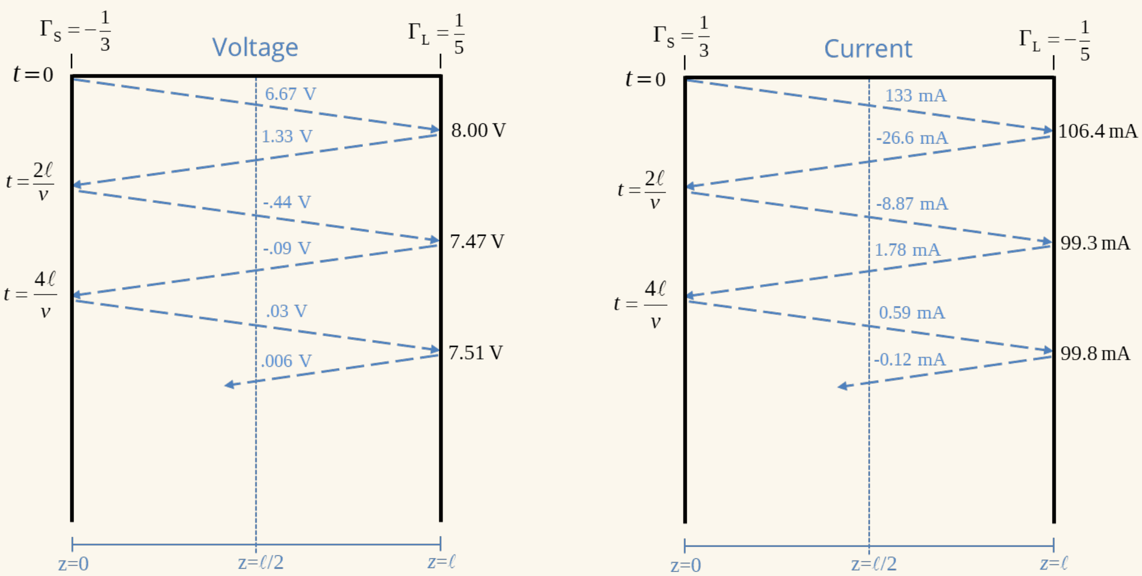
The switch in the circuit shown in the figure below closes at t = 0. Plot the voltage and current as a function of time at z = $\ell/2$.



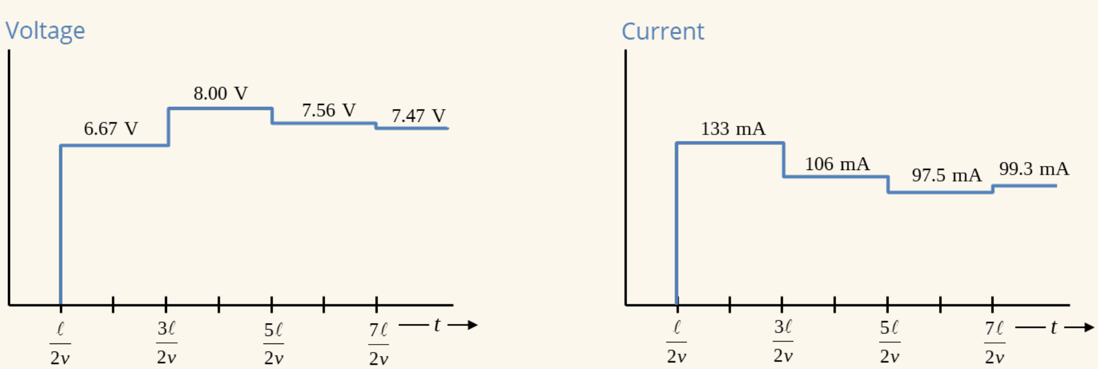
We will start by creating a bounce diagram for the voltage waveform. The voltage reflection coefficient at the source end of the line is $\Gamma_S = \frac{25-50}{25+50} = -\frac{1}{3}$. The voltage reflection coefficient at the load end of the line is $\Gamma_L = \frac{75-50}{75+50} = \frac{1}{5}$. The bounce diagram corresponding to the voltage waveform of the circuit is shown on the left in the figure below. The initial wave launched onto the line has a voltage determined by voltage division to be $V_0 = 10\frac{50}{50+25} = 6.67\,\text{volts}$. The voltage of the first reflected wave is $V_1 = \Gamma_L V_0 = \left(\frac{1}{5}\right)6.67 = 1.33\,\text{volts}$. The second reflected wave has amplitude $V_2 = \Gamma_S V_1 = \left(-\frac{1}{3}\right)1.33 = -0.44\,\text{volts}$. The rest of the wave amplitudes are found in a similar fashion. Although the wave continues to bounce forever, note that the amplitudes become very small after several bounces.

Once the bounce diagram has been completed, we can find the voltage at any position on the line as a function of time. The dashed line in the figure is a line of constant position at the center of the transmission line. The voltage is equal to the sum of the amplitudes of the wave fronts that have passed this position at any time. Therefore, until $t = \ell/2v$, the voltage is zero since no wave fronts have crossed the dashed line. At $t = \ell/2v$, the voltage jumps to 6.67 volts, the amplitude of the first wave front. At $t = 3\ell/2v$, the voltage jumps to 8.0 volts, the sum of the first and second wave fronts. At $t = 5\ell/2v$, the voltage decreases to 7.56 volts, the sum of the first three wave fronts. A plot of the voltage at $z = \ell/2$ as a function of time is shown on the left in the figure at the bottom of this example. As time progresses, we find that the voltage converges to a constant value. This value is 7.5 volts, which is exactly the value that would have been dropped across the load if there were no transmission line. The voltage at all points on the line converges to this steady state value.

In order to plot the current on the line, we create a new bounce diagram. The bounce diagram for the current waveform is shown on the right in the figure below. Note that the current reflection coefficient is equal to the negative of the voltage reflection coefficient. In order to avoid confusion, the symbol Γ will always refer to the voltage reflection coefficient. The amplitude of a current waveform reflection is always equal to –Γ.



The initial current waveform has amplitude, $I_0 = \frac{10}{25+50} = 133\,\text{mA}$. The amplitudes of the reflected waveforms are determined using the same procedure used for the voltage waveforms. A plot of the current at $z = \ell/2$ as a function of time is provided on the right in the figure below. Note that the current converges to a value of 100 mA, which is the value of the current that would be delivered to the load if it had been connected directly to the source.



Voltage and current in the middle of the transmission line.

Pulse Response of a Transmission Line Circuit

Figure 5.11(a) illustrates a transmission line circuit excited by a pulse with amplitude V_o and duration Δt. The pulse can be modeled as the superposition of two step functions. The first step function rises from 0 to V_0 at time t = 0. The second step function falls from 0 to $-V_0$ at time $t = \Delta t$. Viewed in this way; it is relatively straightforward to construct the bounce diagram in Figure 5.11(b). A plot of the voltage at the load end of the line is shown in Figure 5.11(c).

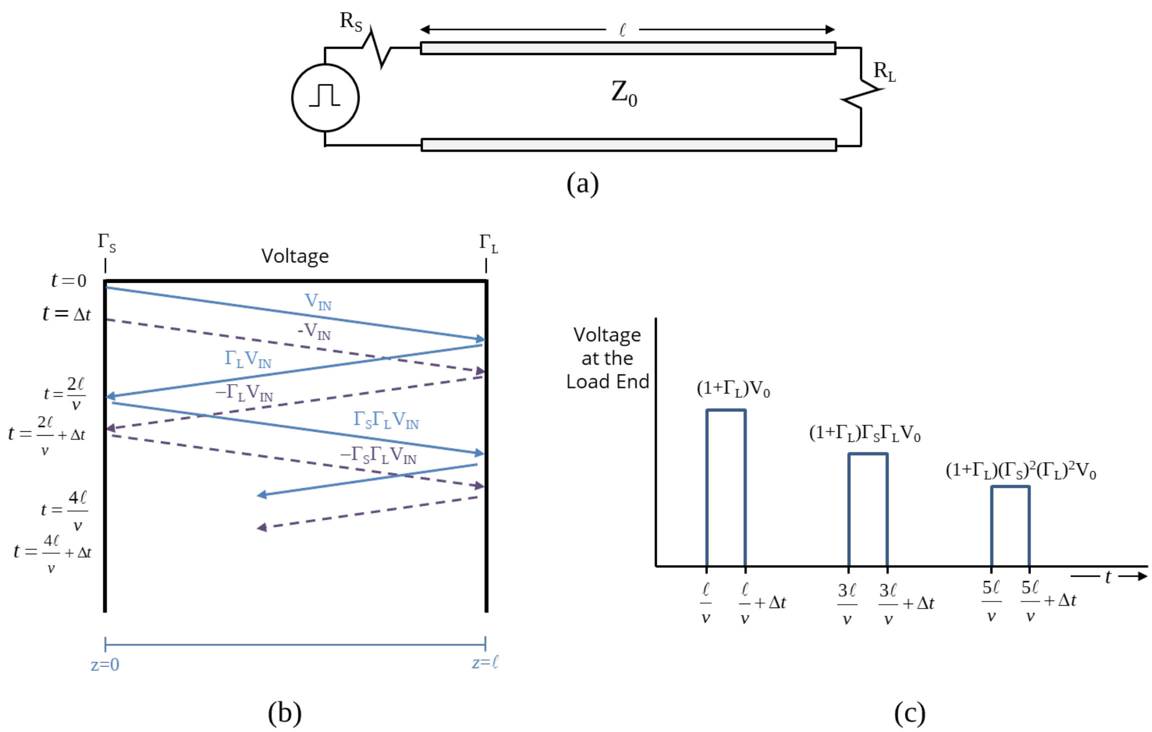


Figure 5.11. Pulse response of a transmission line circuit.

Lossy Transmission Lines

So far, our analysis of the transmission line equations has assumed that the resistance and conductance per unit length are zero (i.e., R=G=0). Adding loss to the transmission line equations complicates their solution considerably. Pulses propagating along the line are both attenuated and distorted. If the loss is small, the distortion may be minimal, and the main effect of the loss may be a slight attenuation.

One method for analyzing lossy transmission lines is to represent the transmission line as a lumped-element circuit (like the one in Figure 5.4) and utilize a SPICE circuit solver. This method yields very accurate results if the length represented by each lumped element section is less than about a tenth of a wavelength at the highest frequency of significance in the modeled waveform.

Another method is to represent the input signal in the frequency domain, calculate the transmission line response, and reconstruct the time-domain signal. As the next section will demonstrate, lossy transmission lines are more straight-forward to analyze in the frequency domain.

Transmission Lines in the Frequency Domain

To analyze the response of a transmission line to a time-harmonic excitation, we must investigate solutions to the transmission line equations in the frequency domain. The lumped element model of a transmission line is shown in Figure 5.12. Phasor notation is used to represent the voltage, *V(z)*, and the current, *I(z)*, at a given frequency, ω. The transmission line equations (5.3) and (5.5) can now be written in phasor form,

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z) \tag{5.20}$$

$$\frac{dI(z)}{dz} = -(G + j\omega C)V(z) \tag{5.21}$$

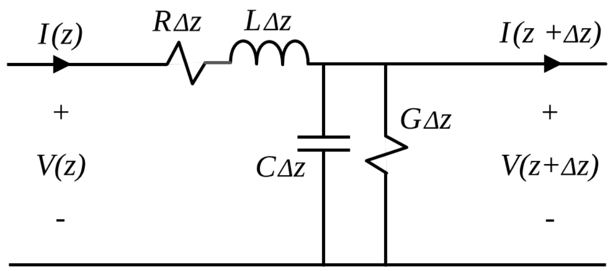


Figure 5.12. Lumped element model of transmission line section.

Taking the derivative of (5.20) with respect to *z* and combining it with (5.21) yields,

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \tag{5.22}$$

where,

$$\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)}\,. \tag{5.23}$$

Similarly, we can combine (5.20) and the derivative of (5.21) with respect to *z* to solve for the current,

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)\,. \tag{5.24}$$

In general, γ is a complex number whose value depends only on the properties of the transmission line,

$$\gamma = \alpha + j\beta \tag{5.25}$$

where α is known as the attenuation constant and β is the propagation constant of the line.

Equations (5.22) and (5.24) are ordinary differential equations with constant coefficients. They are wave equations with solutions of the form,

$$V(z) = V^{+}e^{-\gamma z} + V^{-}e^{+\gamma z} \tag{5.26}$$

$$I(z) = I^{+}e^{-\gamma z} + I^{-}e^{+\gamma z} \tag{5.27}$$

where V^+, V^-, I^+ and I^- are complex constants whose values are determined by the boundary conditions.

We see from Equation (5.26) that the voltage *V(z)* at any point on the transmission line has two components $V^+e^{-\gamma z}$ and $V^-e^{\gamma z}$. If we represent the complex value V^+ as its magnitude and phase, $|V^+|\, e^{j\phi}$, and expand the first component to its full time-domain representation,

$$\begin{aligned} V^{+}e^{-\gamma z} &= \mathrm{Re}\left[V^{+}e^{-(\alpha+j\beta)z}e^{j\omega t}\right] \\ &= \mathrm{Re}\left[\left|V^{+}\right|e^{-\alpha z}e^{j(\omega t-\beta z+\phi)}\right] \\ &= \left|V^{+}\right|e^{-\alpha z}\cos(\omega t-\beta z+\phi) \\ &= \left|V^{+}\right|e^{-\alpha z}\cos\omega\left(t-\frac{z}{v}+\frac{\phi}{\omega}\right), \end{aligned} \tag{5.28}$$

we see that it is of the form, $F(t\text{-}z/v)$. This indicates that $V^+e^{-\gamma z}$ is a forward-traveling sinusoidal wave. Likewise, $V^-e^{\gamma z}$ is a backward-traveling sinusoidal wave. The voltage at any position is a superposition of these two sinusoids.

The term $e^{-\alpha z}$ represents an exponential decay in the amplitude of the forward-traveling wave as it moves in the +*z* direction. The backward traveling wave exhibits a similar, $e^{\alpha z}$, decay in the –*z* direction.

From (5.23) and (5.25), it is clear that the attenuation constant is,

$$\alpha = \mathrm{Re}\left[\gamma\right] = \mathrm{Re}\left[\sqrt{\left(R+j\omega L\right)\left(G+j\omega C\right)}\right]. \tag{5.29}$$

Because it is the square root of a complex number, there is no simple general expression for α in terms of R, L, G and C. However, for most transmission lines, the loss is relatively low at the frequencies of interest (i.e., $R \ll \omega L$ and $G \ll \omega C$). In this case, we can use the low-loss approximation for α,

$$\alpha = \mathrm{Re}\left[\gamma\right] \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right). \tag{5.30}$$

Typically, in cables and circuit board structures below 1 GHz, the conductor losses are greater than the dielectric losses and the attenuation constant is approximately,

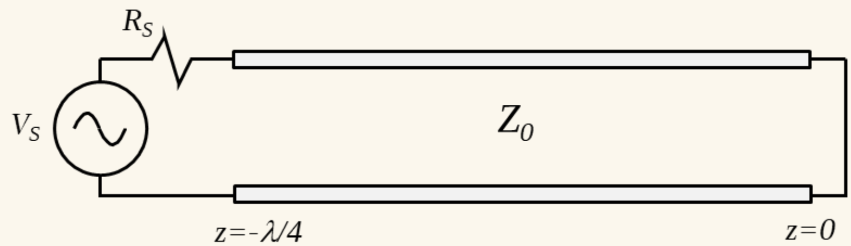
$$\alpha \approx \frac{R}{2Z_0}. \tag{5.31}$$

A wave traveling 1 meter along a transmission line is attenuated by a factor of e^{α}. It is common to express the attenuation of a transmission line in dB/m,

$$\text{attenuation in dB/m} = 20\log e^{\alpha} \simeq 8.7\alpha . \tag{5.32}$$

Example 5-5: Shorted Lossless Transmission Line

Determine the voltage at all points along the shorted lossless transmission line in the figure below.



Note that we have defined $z = 0$ to be at the load end of the line for convenience in doing the calculations. Since the line is lossless, $\alpha = 0$ and $\gamma = j\beta$. The general solution for the voltage on the line (5.26) reduces to,

$$V(z) = V^{+}e^{-j\beta z} + V^{-}e^{+j\beta z}.$$

Applying the boundary condition at $z = 0$,

$$V(0) = 0 = V^{+}e^{0} + V^{-}e^{0}.$$

Therefore, $V^{+}=-V^{-}$ and,

$$V(z) = V^{+}\left[e^{-j\beta z} - e^{+j\beta z}\right].$$

Applying Euler's identity ($e^{j\beta z} - e^{-j\beta z} = 2j \sin \beta z$),

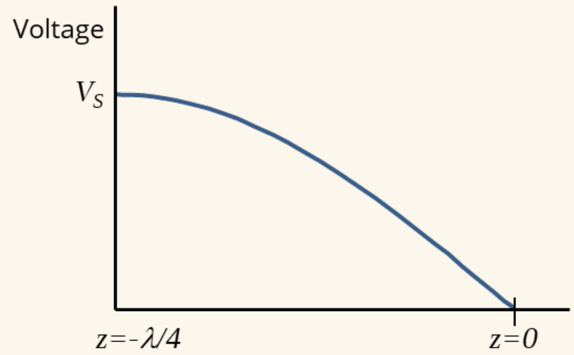
$$V(z) = -2jV^{+}\sin\beta z.$$

Now, we find the value of V^{+} by applying the other boundary condition,

$$V(-\lambda/4) = -2jV^{+}\sin\left(-\beta\lambda/4\right) = V_0$$

since $\beta = 2\pi/\lambda$, $\sin\beta\,\lambda/4 = \sin\pi/2 = 1$ and therefore $V^{+} = \dfrac{V_0}{2j}$. Substituting this value for V^{+} in the above equations, we get an expression for the voltage everywhere on the line,

$$V(z) = V_0 \sin\beta z = V_0 \sin\left(2\pi\frac{-z}{\lambda}\right).$$



This voltage is plotted above. Note that this solution is sinusoidal. It is composed of two components: one forward-traveling and one backward-traveling wave. However, the overall solution does not travel down the transmission line. The load in this example is a short circuit and cannot dissipate any power. Therefore, the time-average power moving in one direction must equal the time-average power moving in the other direction. The overall waveform, *V(z)*, does not propagate down the line. A waveform of this type is called a *standing wave*.

Solving Time-Harmonic Transmission Line Problems

Substituting (5.26) and (5.27) into (5.20) we can show that,

$$-\gamma\left[V^+e^{-\gamma z}-V^-e^{\gamma z}\right]=-\left(R+j\omega L\right)\left[I^+e^{-\gamma z}+I^-e^{\gamma z}\right]. \tag{5.33}$$

Using (5.23), we get,

$$V^+e^{-\gamma z}-V^-e^{\gamma z}=\sqrt{\frac{R+j\omega L}{G+j\omega C}}\left[I^+e^{-\gamma z}+I^-e^{\gamma z}\right]. \tag{5.34}$$

Since this must be true for all *z*,

$$\frac{V^+}{I^+}=Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}} \tag{5.35}$$

$$\frac{V^-}{I^-}=-Z_0=-\sqrt{\frac{R+j\omega L}{G+j\omega C}}\ . \tag{5.36}$$

Z_0 gives us the ratio of the voltage to the current in the forward-traveling component of the wave or the negative of the voltage-to-current ratio in the backward-traveling component of the wave. In general, however, the ratio of the total voltage to the total current on the line is a function of position.

Once again, for convenience, we will define $z = 0$ to be the location of the load. Thus, the load impedance, Z_L is equal to the ratio of the total voltage at $z = 0$ to the total current at $z = 0$,

$$Z_L = \frac{V(0)}{I(0)} = \frac{V^+ + V^-}{I^+ + I^-} = \frac{V^+ + V^-}{V^+/Z_0 - V^-/Z_0} = Z_0 \frac{V^+ + V^-}{V^+ - V^-} . \tag{5.37}$$

Rearranging the terms in (5.37) yields,

$$\frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0} . \tag{5.38}$$

As in the time-domain case, we define the voltage reflection coefficient to be the ratio of the reflected wave to the incident wave,

$$\Gamma_L = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0} . \tag{5.39}$$

Comparing this result to Equation (5.19), we see that Γ_L is calculated from the load impedance and the characteristic impedance in the same way whether we are working in the time domain or the frequency domain. In the frequency domain, however, the quantities Z_0 and Z_L are complex quantities and therefore Γ_L may also be complex.

Lossless Transmission Lines in the Frequency Domain

Many practical transmission lines have relatively low loss and can be modeled as lossless (i.e., $R \approx G \approx 0$). Lossless lines have a real characteristic impedance,

$$Z_0 = \sqrt{L/C} \tag{5.40}$$

and an imaginary propagation constant,

$$\gamma = -j\beta = j\omega\sqrt{LC} . \tag{5.41}$$

The voltage along a lossless line is given by,

$$V(z) = V^+ \left[e^{-j\beta z} + \Gamma_L e^{j\beta z} \right] . \tag{5.42}$$

The current along a lossless line is given by,

$$I(z) = I^+ \left[e^{-j\beta z} - \Gamma_L e^{j\beta z} \right] . \tag{5.43}$$

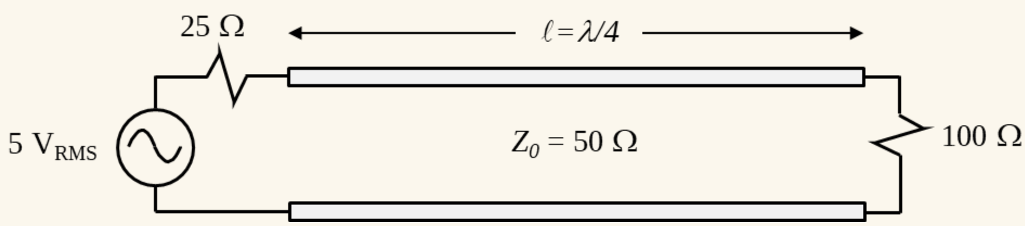
The impedance at $z=-\ell$ is found by combining Equations (5.42) and (5.43) with Equations (5.35) and (5.39),

$$Z(-\ell) = \frac{V(-\ell)}{I(-\ell)} = Z_o \frac{Z_L \cos\beta\ell + jZ_o \sin\beta\ell}{Z_o \cos\beta\ell + jZ_L \sin\beta\ell} = Z_o \frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell} . \tag{5.44}$$

Example 5-6: Average Power Delivered to Load

Calculate the average power delivered to a 100-Ω load connected to a 5.0-volt, 25-Ω source through a quarter-wavelength 50-Ω transmission line.

The configuration described in this example is illustrated below. There are a couple of approaches that can be used to solve this problem; however, the most straight-forward method is to solve for the impedance at the input to the transmission line, then find the average power dissipated in this impedance. Since the transmission line is lossless, all the power delivered to the input of the transmission line is dissipated in the load impedance.



The impedance at the input to the transmission line is found using Equation (5.44),

$$Z(-\lambda/4) = 50\frac{100\cos\beta\,\lambda/4 + j50\sin\beta\,\lambda/4}{50\cos\beta\,\lambda/4 + j100\sin\beta\,\lambda/4} = 50\frac{100\cos\pi/2 + j50\sin\pi/2}{50\cos\pi/2 + j100\sin\pi/2} = 25\ \Omega\ .$$

Now the problem has been reduced to finding the power delivered to a 25-Ω load from a 5.0-volt, 25-Ω source,

$$P_{AVE} = |I|^2 R = \left|\frac{5.0}{25+25}\right|^2 25 = 250\ \text{mW}\ .$$

Short Transmission Line Segments

Brief disruptions in a controlled-impedance signal path (e.g., due to electrical connections or layer transitions) can often be modeled as short transmission lines that have a characteristic impedance different from the target impedance. When the disruptions are electrically short, they can usually be modeled using a single inductor or capacitor. In this section, we explore how to determine the appropriate value of the lumped elements used to model these types of disruptions.

Consider the model for a discontinuity in a matched transmission line shown in Figure 5.13. If we assume that the values of Z_{01} and Z_{02} are real numbers (i.e., the transmission lines are low-loss), then the termination of the transmission line segment on the left is simply the input impedance of the transmission line in the middle terminated by Z_{01} as shown in Figure 5.14.

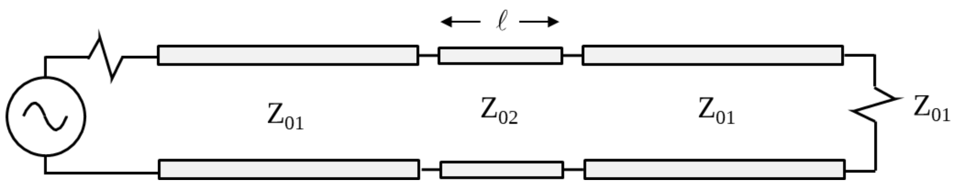


Figure 5.13. Model of a discontinuity in a matched transmission line.

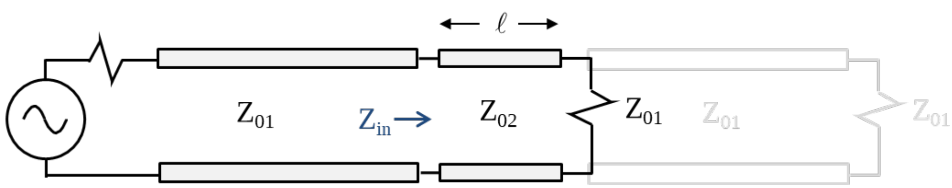


Figure 5.14. Simplified model of discontinuity.

The general expression for the input impedance of the transmission line representing the discontinuity is,

$$Z_{in} = Z_{02} \frac{Z_{01} + jZ_{02} \tan \beta \ell}{Z_{02} + jZ_{01} \tan \beta \ell}. \tag{5.45}$$

where β is the propagation constant of the transmission line ($\beta = 2\pi/\lambda$). If the length of the discontinuity, ℓ, is small relative to the signal wavelength, then $\beta\ell$ is a small number and tan $\beta\ell$ is approximately equal to $\beta\ell$. In this case, (5.45) reduced to,

$$Z_{in} \approx Z_{02} \frac{Z_{01} + jZ_{02} \beta \ell}{Z_{02} + jZ_{01} \beta \ell}. \tag{5.46}$$

Solving for the real and imaginary parts of Z_{in},

$$\begin{aligned} Z_{in} &\approx Z_{02} \frac{Z_{01} + jZ_{02} \beta \ell}{Z_{02} + jZ_{01} \beta \ell} \left[\frac{Z_{02} - jZ_{01} \beta \ell}{Z_{02} - jZ_{01} \beta \ell} \right] \\ &\approx Z_{02} \frac{Z_{01}Z_{02} + Z_{01}Z_{02} (\beta\ell)^2}{(Z_{02})^2 + (Z_{01}\beta\ell)^2} + jZ_{02} \frac{(Z_{02})^2 \beta\ell - (Z_{01})^2 \beta\ell}{(Z_{02})^2 + (Z_{01}\beta\ell)^2}. \end{aligned} \tag{5.47}$$

Since $\beta\ell$ is small (i.e., $\beta\ell << 1$), (5.47) reduces to,

$$Z_{in} \approx Z_{01} + j\beta\ell Z_{02} \left[1 - \left(\frac{Z_{01}}{Z_{02}} \right)^2 \right]. \tag{5.48}$$

Now let's consider two cases: $Z_{02} \geq Z_{01}$ and $Z_{02} \leq Z_{01}$. If $Z_{02} \geq Z_{01}$ (i.e., the characteristic impedance of the discontinuity is greater than or equal to the characteristic impedance of the transmission line), then (5.48) can be written as,

$$\begin{aligned} Z_{in} &\approx Z_{01} + j\beta\ell Z_{02}\left[1-\left(\frac{Z_{01}}{Z_{02}}\right)^2\right] \\ &\approx Z_{01} + j\omega\frac{Z_{02}}{v}\ell\left[1-\left(\frac{Z_{01}}{Z_{02}}\right)^2\right]. \\ &\approx Z_{01} + j\omega L\ell\left[1-\left(\frac{Z_{01}}{Z_{02}}\right)^2\right] \end{aligned} \tag{5.49}$$

where $L\ell$ is the inductance per unit length of the discontinuity times the length. In other words, when the discontinuity has a characteristic impedance greater than the termination resistance, it can be modeled as a lumped inductor in series with the termination resistor as illustrated in Figure 5.15.

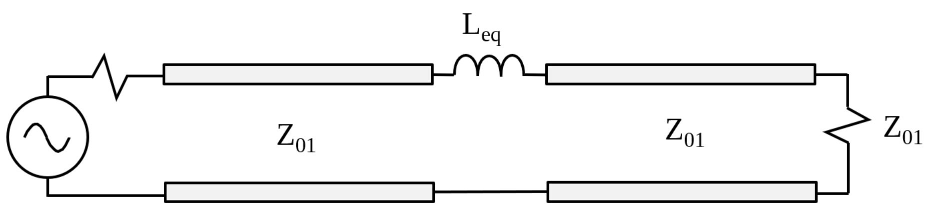


Figure 5.15. Simplified model of short discontinuity when $Z_{02}>Z_{01}$.

The value of the equivalent inductance is,

$$L_{eq} \approx L\ell\left[1-\left(Z_{01}/Z_{02}\right)^2\right]. \tag{5.50}$$

For termination resistances that are much smaller than the characteristic impedance, the lumped inductance is approximately equal to the inductance per unit length times the length of the transmission line. As the termination resistance approaches the value of the characteristic impedance the equivalent inductance decreases. As expected, there is no equivalent inductance when the discontinuity is terminated with a matched resistance.

Now let's consider the case where an electrically short transmission line is terminated with a resistance higher than the characteristic impedance, $Z_{02} \leq Z_{01}$. In this case, it's more convenient to solve for the input admittance (i.e., the inverse of the input impedance),

$$\begin{aligned} Y_{in} &= \frac{1}{Z_{in}} = \frac{1}{Z_{02}}\frac{Z_{02}+jZ_{01}\beta\ell}{Z_{01}+jZ_{02}\beta\ell} \\ &= \frac{1}{Z_{02}}\frac{Z_{02}+jZ_{01}\beta\ell}{Z_{01}+jZ_{02}\beta\ell}\left[\frac{Z_{01}-jZ_{02}\beta\ell}{Z_{01}-jZ_{02}\beta\ell}\right] \\ &= \frac{1}{Z_{02}}\frac{Z_{01}Z_{02}-Z_{01}Z_{02}\left(\beta\ell\right)^2}{\left(Z_{01}\right)^2+\left(Z_{02}\right)^2\left(\beta\ell\right)^2} + j\frac{1}{Z_{02}}\frac{\left[\left(Z_{01}\right)^2-\left(Z_{02}\right)^2\right]\left(\beta\ell\right)}{\left(Z_{01}\right)^2+\left(Z_{02}\right)^2\left(\beta\ell\right)^2}. \end{aligned} \tag{5.51}$$

Applying the constraints, $Z_{02} \le Z_{01}$ and $\beta\ell << 1$, we get,

$$\begin{aligned} Y_{in} &\approx \frac{1}{Z_{01}} + j\frac{\beta\ell}{Z_{02}}\left[1-\left(Z_{02}/Z_{01}\right)^2\right] \\ &\approx \frac{1}{Z_{01}} + j\omega C\ell\left[1-\left(Z_{02}/Z_{01}\right)^2\right]. \end{aligned} \tag{5.52}$$

In other words, the real part of the input admittance is the same as the admittance of the termination and the imaginary part of the input admittance is the same as a capacitor have a value,

$$C_{eq} \approx C\ell\left[1-\left(Z_{02}/Z_{01}\right)^2\right]. \tag{5.53}$$

Therefore, for $Z_{02} < Z_{01}$, we can model the discontinuity as a lumped capacitance as shown in Figure 5.16.

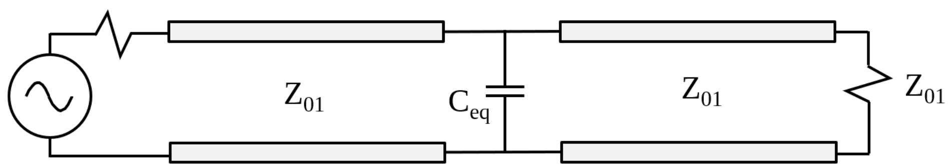


Figure 5.16. Simplified model of short discontinuity when $Z_{01}>Z_{02}$.

Note that the equivalent circuits in Figures 5.15 and 5.16 apply to any short transmission line segment terminated with a resistance, not just discontinuities in a longer transmission line. A short transmission line segment terminated in a resistance lower than its characteristic impedance can be modeled as a lumped inductor in series with the termination resistor. A short transmission line segment terminated in a resistance higher than its characteristic impedance can be modeled as a lumped capacitor in parallel with the termination resistor.

If the termination resistance is much lower than the characteristic impedance, the value of the lumped inductor is the total inductance of the transmission line. As the termination resistance approaches the value of the characteristic impedance, the value becomes somewhat smaller than the total inductance of the line.

If the termination resistance is much higher than the characteristic impedance, the value of the lumped capacitor is the total capacitance of the transmission line. As the termination resistance approaches the value of the characteristic impedance, the value becomes somewhat smaller than the total capacitance of the line. If the termination impedance equals the characteristic impedance, the input impedance of the transmission line segment is equal to the termination resistance.

The Time-Domain Reflectometer

Ideally, a matched transmission line has a constant impedance from the signal source to the signal termination. In real systems, however, interconnections, parasitic coupling, and design tolerances contribute to variations in the impedance that can cause some of the signal power to be reflected back towards the source. The time-domain reflectometer (TDR) is a measurement instrument designed to locate and quantify unintended reflection sources.

A TDR works by launching a step change in the voltage across the input to a transmission line and monitoring the voltage at that point. If the line has no discontinuities and is perfectly matched, no change in the voltage at the input will be observed after the initial step. However, any discontinuity in the transmission line's impedance will reflect some of the signal back to the source where it can be recorded.

Figure 5.17 illustrates a TDR connected to a 50-Ω transmission line with a 10-Ω discontinuity located a distance, ℓ, from the source end. The measurement of the input voltage commences with the closing of the switch at t=0 and is plotted in Figure 5.18. The vertical scale in this plot is offset by $V_S/2$ and normalized. The relative voltage plotted is,

$$\text{Normalized Voltage} = \frac{V_{IN} - {V_S}/{2}}{{V_S}/{2}} = 2\left(\frac{V_{IN}}{V_S}\right) - 1\,. \tag{5.54}$$

This unitless value is equal to the amplitude of the reflection coefficient.

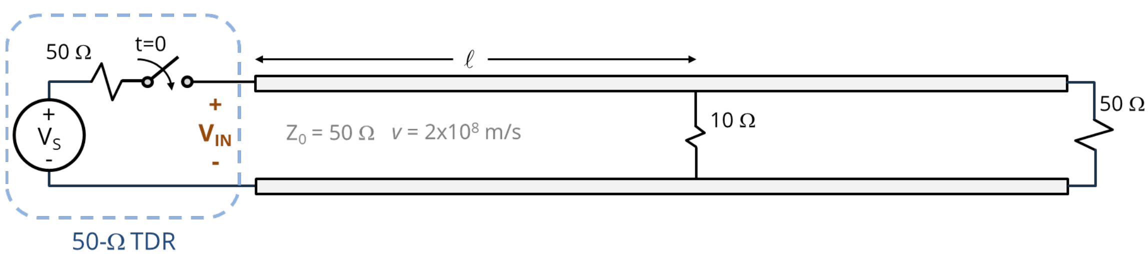


Figure 5.17. TDR connected to a transmission line with a 10-Ω shunt discontinuity.

The plot in Figure 5.18 shows that the initial normalized voltage is zero, corresponding to a measured voltage of $V_S/2$. This is what we would expect from a 50-Ω source that initially sees only the 50-Ω characteristic impedance of the transmission line.

The voltage step propagates down the line until it reaches the 10-Ω resistor. At that point, the wave sees the 10-Ω resistance in parallel with the 50-Ω impedance of the remaining transmission line. The parallel combination of these resistances is 8.33 Ω. The sudden change in impedance produces a reflected voltage step with an amplitude,

$$V_{reflected} = \Gamma V_{incident} = \frac{Z_L - Z_0}{Z_L + Z_0}\left(\frac{V_S}{2}\right) = \frac{8.33-50}{8.33+50}\left(\frac{V_S}{2}\right) = -0.766\left(\frac{V_S}{2}\right). \tag{5.55}$$

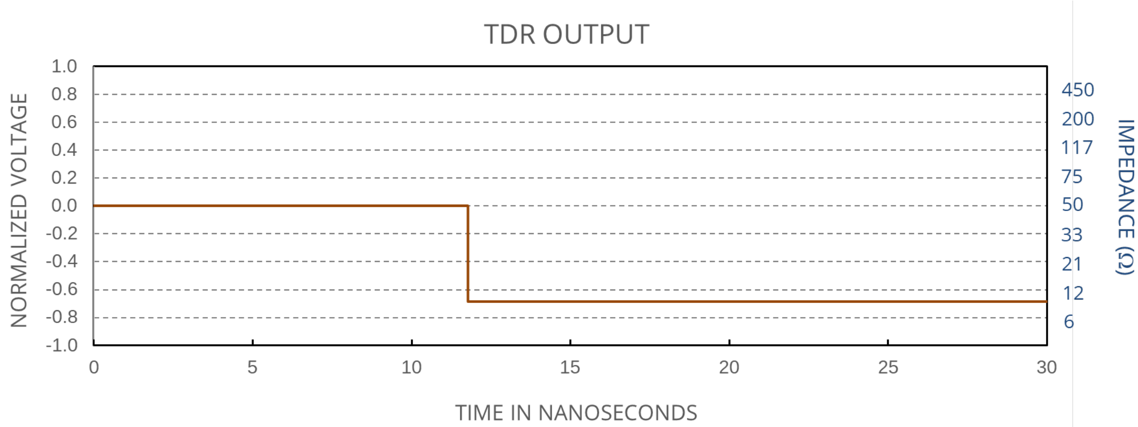


Figure 5.18. TDR output for a 10-Ω shunt discontinuity.

After 12 ns, the reflected wave reaches the TDR and the voltage takes on a new value,

$$V_{IN} = V_{incident} + V_{reflected} = \frac{V_S}{2} - 0.766\left(\frac{V_S}{2}\right) = 0.234\left(\frac{V_S}{2}\right). \tag{5.56}$$

Using (5.54) this value of V_{IN} translates to a normalized value of -0.766, which is the reflection coefficient at the discontinuity.

If we know the reflection coefficient at the discontinuity, we can determine the impedance of the discontinuity using the equation,

$$Z_L = Z_0 \frac{1+\Gamma}{1-\Gamma}. \tag{5.57}$$

In this way, we can map any value of reflection coefficient in our 50-Ω system to a resistance. This mapping is indicated by the vertical scale on the right side of the plot in Figure 5.18.

The plot in Figure 5.18 not only indicates the impedance at the discontinuity, but also its location. Note that the step in the voltage occurs after 12 ns. This is the amount of time it took for the wave to propagate from the TDR to the discontinuity and back. Since the velocity of propagation in the line is 2×10^8 m/s (as indicated in Figure 5.17), the distance to the discontinuity is,

$$\ell = \frac{1}{2}\left(2\times10^{8} \text{ m/s}\right)\left(12\times10^{-9} \text{ s}\right) = 1.2 \text{ m}. \tag{5.58}$$

Figure 5.19 shows another transmission line configuration with a 50-Ω discontinuity in series with the conductors. The TDR output corresponding to this configuration is shown in Figure 5.20. The overall impedance at the discontinuity is 100 Ω and the 15 ns round trip propagation time indicates it is 1.5 m from the TDR.

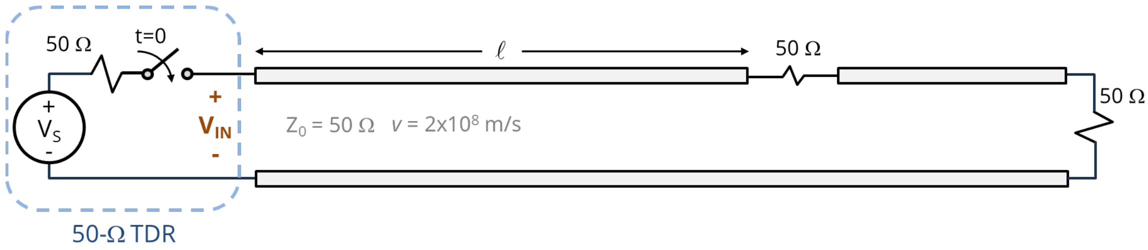


Figure 5.19. A series 50-Ω discontinuity in a transmission line.

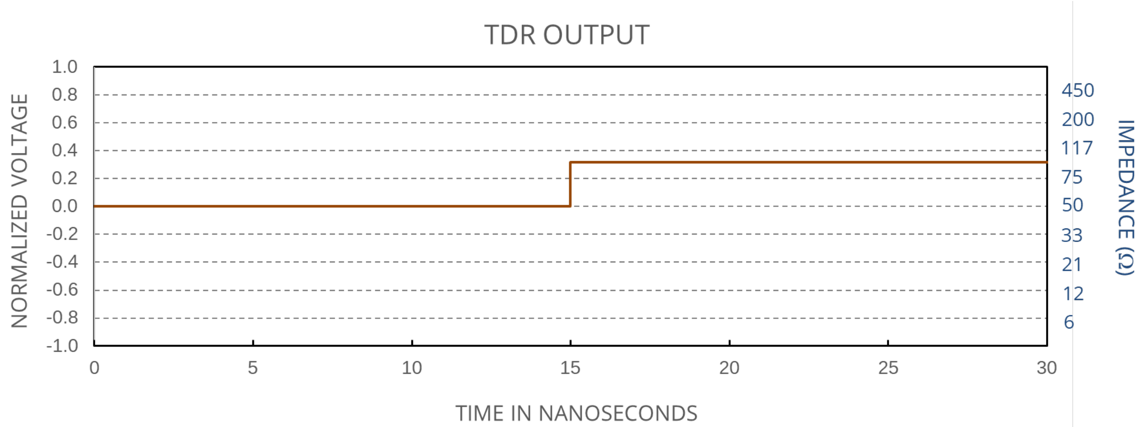


Figure 5.20. TDR output for a series 50-Ω discontinuity.

TDRs can also be used to detect and identify reactive elements such as inductances and capacitances. Figure 5.21 shows a transmission line with an inductive discontinuity (e.g., from a mismatched connector). The corresponding TDR output is shown in Figure 5.22.

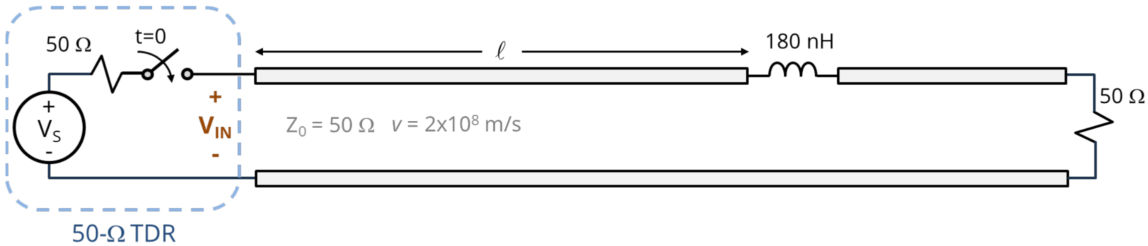


Figure 5.21. An inductive discontinuity in a matched transmission line.

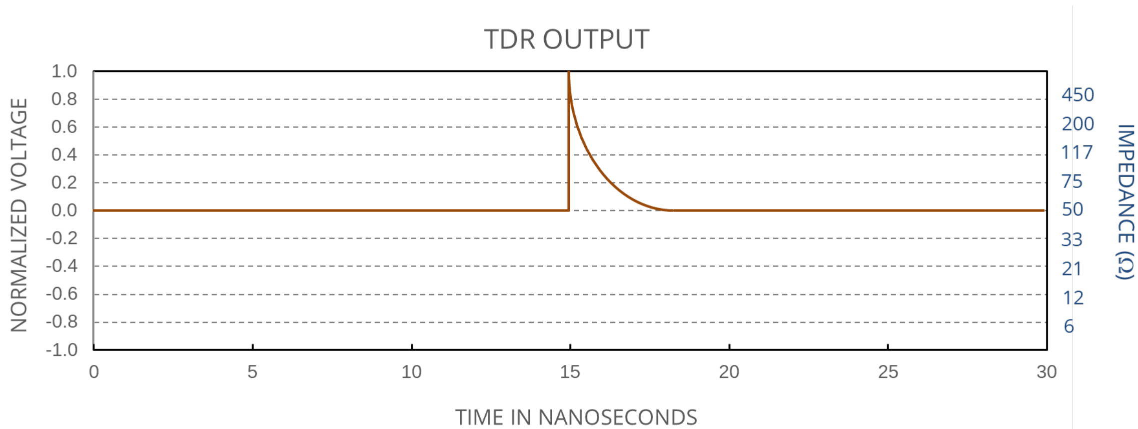


Figure 5.22. TDR output for an inductive discontinuity.

Note that at the instant the step reaches the inductance, it behaves like an open circuit with a reflection coefficient of +1. In the steady state, the inductance looks like a short circuit so the impedance at the discontinuity is 50 Ω and the reflection coefficient is zero. Since the RL circuit forming the discontinuity is a first-order circuit, the transition from the initial voltage to the steady-state voltage is exponential with a transition time equal to,

$$t_r = 2.2\frac{L}{R} = 2.2\frac{180\times 10^{-9}\text{ H}}{50\ \Omega + 50\ \Omega} \approx 4\text{ ns}. \tag{5.59}$$

Figure 5.23 shows the TDR output for another discontinuity. Note that initially it looks like a short circuit with a reflection coefficient of -1. In the steady state, the discontinuity is gone, and the impedance is that of a matched termination. This is consistent with the behavior expected from a shunt capacitor.

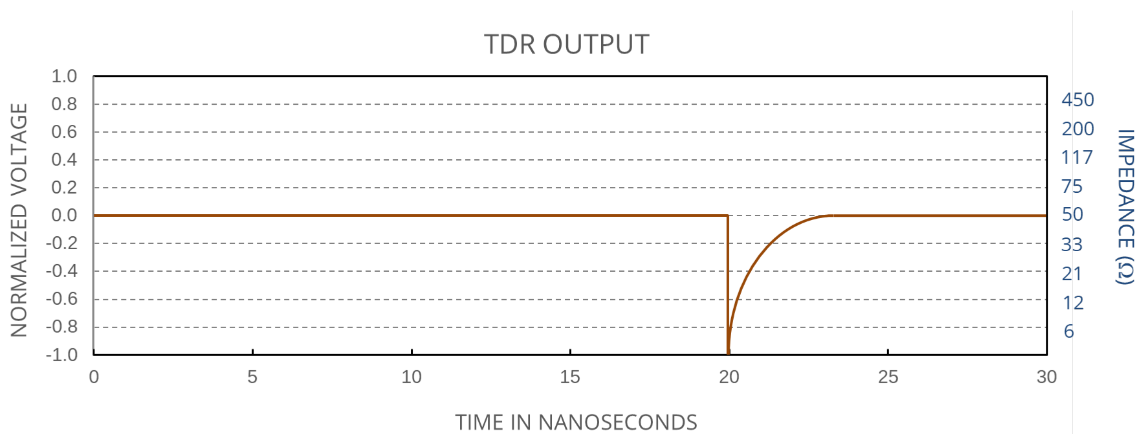


Figure 5.23. Plot of V_{IN} as a function of time.

In this case, the 20-ns round trip delay indicates the capacitance is 2.0 meters from the TDR. The 4-ns transition time indicates that the value of the capacitor is,

$$t_r = 2.2RC \quad \Rightarrow \quad C = \frac{t_r}{2.2R} = \frac{4\times 10^{-9}\text{ s}}{2.2\left(50\ \Omega \| 50\ \Omega\right)} \approx 73\text{ pF}. \tag{5.60}$$